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(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. FIRST SEMESTER EXAMINATION, DECEMBER 2019

FIRST YEAR [BATCH 2019-21]

Date : 12/12/2019 **MATHEMATICS**

Time : 11 am - 1 pm **Paper : I [Abstract Algebra - I]** Full Marks : 50

[Answer all the questions. Maximum you can score is 50.]

1. Let G be a semigroup with identity. Show that G is a group if and only if for all $a \in G$ there exists a unique $b \in S$ such that $aba = a$. [4]
2. Prove that $(\mathbb{Q},+)$ is not cyclic. [1]
3. Let G be a finite group. Show that if G has exactly one nontrivial subgroup, then the order of G is p^2 for some prime p. [5]
4. Let H and K be two subgroups of a finite group G such that $|H| > \sqrt{|G|}$ *and* $|K| > \sqrt{|G|}$. Show that $|H \cap K| > 1$. [1]
5. Let H be a normal subgroup of a finite group G. Let P be a Sylow p-subgroup of H and $N_G(P) = \{g \in G : gPg^{-1} = P\}$. Prove that $G = HN_G(P)$. [4]
6. Prove that there are only two groups of order 6 upto isomorphism. [5]
7. Let G be a group of order 455. Show that G is cyclic. [5]
8. Let G be a group of order 1045. Show that a Sylow 19-subgroup of G is in the centre of G and a Sylow 11-subgroup of G is a normal subgroup of G. [5]
9. Let R be a ring with 1. If a is a nilpotent element of R, prove that $1-a$ and $1+a$ are units. [2]
10. Find all subfields of the field $\mathbb{Z}_p$, where p is a prime number . [2]
11. Let R be a ring with 1. Prove that R has no nontrivial left ideals if and only if R has no nontrivial right ideals. [3]
12. Are the rings $\mathbb{Z}_6$ *and* $\mathbb{Z}_3 \times \mathbb{Z}_2$ isomorphic? Justify. [2]
13. Prove that any integral domain D can be embedded in a field. [5]
14. Find all the units of $\mathbb{Z}[x]$. [3]
15. Prove that in $\mathbb{Z}_8[x], [4]x^2 + [2]x + [4]$ is a zero divisor. [2]
16. Find all maximal and prime ideals of $\mathbb{Z}_{10}$. [4]
17. Show that the polynomial $x^5 + x^2 + [1]$ is irreducible in $\mathbb{Z}_2[x]$. [2]
18. Show that the integral domain $\mathbb{Z}[\sqrt{10}] = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\}$ is FD. [3]
19. Show that 3 is not a prime element in $\mathbb{Z}[i\sqrt{5}]$. [2]

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